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Relating CKM Matrix Parametrizations and Unitarity Triangle

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Abstract—*We attempt to study different parametrizations of CKM matrix and unitarity triangle with the help of the standard model global fit. We calculate the constraint imposed on the unitarity triangle as allowed by the various experiments. The calculated results are fairly in agreement with the experimental data.*

1. INTRODUCTION: CKM MATRIX AND UNITARITY TRIANGLE

The weak force is responsible for the decay of unstable matter particles which are composed of heavy quarks and antiquarks into particles made of their lighter cousins. For examples: $(a)\, n \longrightarrow p + e^- + \overline{v}_e$ $(b)\, \mu^+ \longrightarrow e^+ + \overline{v}_\mu + v_e$ $(c)\, \pi^+ \longrightarrow \mu^+ + v_\mu$. The rates of these decay processes can be related to a set of numbers called the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements [**1**]. The numbers are often shown in a graphical form called the unitarity triangle (Fig. 1). The CKM matrix gives rise to CP violation, the subtle difference between matter and antimatter. The area of the triangle is a measure of the amount of CP violation caused by the weak force. This CP violation partly explains why we live in a matter-dominated universe rather than one full of antimatter or radiation. At Brookhaven National Laboratory led by James Cronin and Val Fitch CP violation was first observed in 1964 in neutral kaon decays-called the short-lived K_S and the longer-lived K_L [**2**].

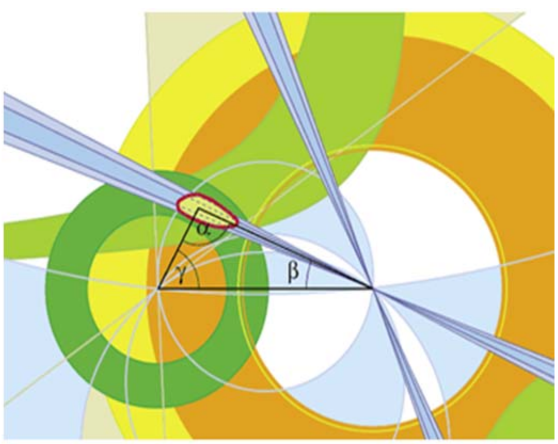

Fig. 1: ***Unitarity triangle***

In a CP-conserving world: $K_s \longrightarrow \pi\,\pi$ & $K_L \longrightarrow \pi\,\pi\,\pi$, while in a CP-violating world K_L was also found to decay to two pions about 0.1% of the time. This manifestation of CP violation is described by the parameter ε_K, and measurements of it constrain the peak of the triangle to lie somewhere on the light green, boomerang-shaped region.

The angle β: By studying the interference between the decays of neutral B^o mesons$(\overline{b}d)$ and their antiparticles $\overline{B}^o(b\overline{d})$, physicists determine the angle β of the untarity triangle. It is the "golden measurement" for the two B factories, BaBar at SLAC in California (**3**), and Belle at KEK in Japan (**4**). Because what they measure is $sin(2\beta)$, there are actually four possible solutions for the angle, which is shown as the four blue jets radiating from the bottom-right corner of the triangle. **The other angles α and γ:** These two angles are more difficult to measure than β. Quantum effects, so-called "penguin" processes, interfere with direct measurements of α, while measurements of γ require studies of rare decay processes. The B factories have been able to measure these angles, adding two constraints to the triangle. Like β, the angle α produces four allowed regions for the upper vertex, shown as blue arc-shaped areas. The angle γ constrains the location of the peak of the triangle to the two wedges radiating from the bottom-left corner of the triangle.

The triangle's left side: By measuring the rates at which bottom quarks decay into up and charm quarks, i.e. $b \longrightarrow u + c$, physicists determine how elements of the CKM matrix called V_{ub} and V_{ub}. The ratio of these two elements $\left(\frac{V_{ub}}{V_{ub}}\right)$ gives the length of the left side of the triangle (fig.1), the end of which must lie in the dark green circle. Many experiments have measured V_{ub} and V_{ub} and the current best measurements come from the B factories at SLAC and KEK.

The triangle's right side: The B^o meson can spontaneously turn into a $\overline{B}^o$ meson, its antiparticle. The rate at which this transformation occurs has been measured by a number of experimets and constrains the length of the right side of the unitarity triangle (fig.1), placing its end in the yellow ring (largely covered by the orange ring). Studying the B_s meson, which contains an anti-bottom quark and a strange quark, and finding no evidence for its transformation into the antiparticle $\overline{B}_s$, i.e. $B_s \nrightarrow \overline{B}_s$, physicists know that the end point of the right side must lie in the orange ring only(fig.1). Experiments at Fermilab are currently improving the measurement, and may soon observe $B_s \longrightarrow \overline{B}_s$ mixing for the first time.

2. PARAMETRIZATIONS OF CABIBBO-KOBAYASHI-MASKAWA (CKM) MATRIX

The quark mass eigenstates differ from the weak eigenstates and the mixing between different quarks is given by the complex Cabibbo-Kobayashi-Maskawa matrix. The matrix elements relate eigenstates of mass and weak interaction, as

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \ or\ V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \quad (1)$$

Here, the element V_{ij} specifies the coupling of the charged currents to the quarks with flavour i and j. It is assumed that the CKM matrix is unitary, and as we know that for a unitary matrix V; $V^{\intercal}V \equiv VV^{\intercal} \equiv \mathrm{I}$ and $V^{\intercal} \equiv V^{-1}$.

Original Kobayashi- Maskawa parameterization: Many parametrizations of the CKM matrix have been proposed till date. We begin with the Original Kobayashi- Maskawa parameterization where there are only three generalized Cabibbo angles, θ_1, θ_2 and θ_3 which one recognizes as so-called Euler angles, and a phase factor δ.

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix},$$

where c_i represents $cos\theta_i$ and s_i represents $sin\theta_i$. The matrix is again unitary,

$$V^{\intercal} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & -s_3 \\ 0 & s_3 & c_3 \end{pmatrix} \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} = V^{-1} \quad (2)$$

$$V = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & -c_1 s_2 c_3 - c_2 s_3 e^{i\delta} & -c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix} (3)$$

In the limit $\theta_2 = \theta_3 = 0$, the third generation decouples, and the usual Cabibbo mixing of the first two generations is recovered. One can identify θ_1 with the Cabibbo angle.

Standard parametrization: The Standard Parameterization of V was proposed by Chau and Keung [**5**] and is advocated by the Particle Data Group (PDG) [**6**]. It is obtained by the product of three (complex) rotation matrices, where the rotations are characterized by the Euler angles θ_{12}, θ_{13} and θ_{12}, which are the mixing angles between the generations, and one overall phase δ

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} . (4)$$

With $c_{ij} = cos\theta_{ij}$ and $s_{ij} = sin\theta_{ij}\ for\ i < j = 1,2,3.$ This parametrization is strictly satisfies the unitarity relation $V^{\intercal}V \equiv VV^{\intercal} \equiv \mathrm{I}$.

Wolfenstein parametrization: Following the observation of a hierarchy between the different matrix elements, Wolfenstein [**7**] proposed an expansion of the CKM matrix in terms of the four parameters λ, A, ρ and η (λ ≃|Vus| ∼ 0.22 being the expansion parameter), which is widely used in contemporary literature, and which is the parameterization employed in this work and in CKMfitter. He parametrized the standard representation for better results and more accurate generalization and are more transparent than the standard parametrization and also allow a fast estimate of different contributions to a given decay amplitude. We make the following change of variables in the standard parametrization (4) to all orders of λ [**8**]

$$s_{12} = \lambda, s_{23} = A\lambda^2, s_{13}e^{-i\delta} = A\lambda^3(\rho - i\eta), \quad (5)$$

Finally (4) reduces to

$$V_{CKM} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4 & \lambda + O(\lambda^7) & A\lambda^3(\rho - i\eta) \\ -\lambda + \frac{1}{2}A^2\lambda^5[1 - 2(\rho + i\eta)] & 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4(1 + 4A^2) & A\lambda^2 + O(\lambda^8) \\ A\lambda^3(1 - \overline{\rho} - i\overline{\eta}) & -A\lambda^2 + \frac{1}{2}A\lambda^4[1 - 2(\rho + i\eta)] & 1 - \frac{1}{2}A^2\lambda^4 \end{pmatrix} (6)$$

Where terms $O(\lambda^6)$ and higher order terms have been neglected. A non-vanishing η is responsible for CP-violation in the minimal flavour violation.

The Jarlskog Invariant: The phase-convention independent measurement of CP violation, J, shown by Jarlskog is given by

Im

$$[V_{ij}V_{kl}V_{il}^*V_{kj}^*] = J \sum_{m,n=1}^{3} \varepsilon_{ikm}\, \varepsilon_{jin,} \quad (7)$$

where V_{ij} are the CKM matrix elements and ε_{ikm} is the total antisymmetric tensor. One representation of Eq. (7) reads, for instance, J = Im $[V_{ud}V_{cs}V_{us}^*V_{cd}^*]$. A non-vanishing CKM phase and hence CP violation necessary requires $J \neq 0$. The Jarlskog parameter expressed in the Standard Parameterization (4) reads

$$J = c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13}sin\delta. \quad (8)$$

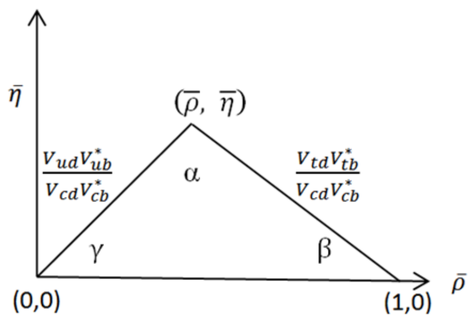

Fig. 2: The rescale Unitarity Triangle in the Wolfenstein parameterization.

and, using the Wolfenstein parameterization, one finds

$$\mathrm{J} = A^2\lambda^6\eta\left(1-\frac{\lambda^2}{2}\right) + \mathcal{O}(\lambda^{10}) \sim 10^{-5}. \tag{9}$$

The empirical value of J is small compared to its mathematical maximum of 1/(6√3) ≃ 0.1 showing that CP violation is suppressed as a consequence of the strong hierarchy exhibited by the CKM matrix elements. Remarkably, to account for CP violation requires not only a non-zero J but also a non-degenerated quark-mass hierarchy. Equal masses for at least two generations of up-type or down-type quarks would eliminate the CKM phase.

3. CALCULATIONS OF CKM ELEMENTS

We will use the results of the four wolfenstein parameters namely $A, \lambda, \bar{\rho}$ and $\bar{\eta}$ from the "SM Global Fit" experiment [**9**] The values are:

$$A = 0.810^{-0.018}_{-0.024}, \lambda = 0.22548^{+0.00068}_{-0.00034}, \bar{\rho} = 0.145^{-0.013}_{-0.007}, \bar{\eta} = 0.343^{-0.011}_{-0.012}.$$

$$V_{ud} = 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4$$

Now, for $\lambda = 0.22548 + 0.00068 = 0.22616, V_{ud} = 0.9741.$

for $\lambda = 0.22548 - 0.00034 = 0.22514, V_{ud} = 0.9744.$

Therefore, $V_{ud} = 0.9741\ to\ 0.9744. V_{cs} = 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4(1+4A^2)$

For $\lambda = 0.22616$, and $A = 0.810 - 0.018 = 0.792, V_{cs} = 0.9733.$

For $\lambda = 0.22616$, and $A = 0.810 - 0.024 = 0.786, V_{cs} = 0.9734.$

Therefore, $V_{cs} = 0.9733\ to\ 0.9734$. Similarly, we find the other elements, neglecting the phase as:

$$V_{tb} = 0.9986\ to\ 0.9989, V_{cd} = -0.2245\ to - 0.2242\ , V_{us} = 0.2252\ to\ 0.2257, V_{ub} = 0.0035\ to\ 0.0037, V_{cb} = 0.04185\ to\ 0.4184, V_{ts} = 0.0410\ to\ 0.0411\ , V_{td} = 0.0005\ to\ 0.0087.$$

The matrix becomes:

$$V_{CKM} = \begin{pmatrix} 0.9741\ to\ 0.9744 & 0.2252\ to\ 0.2257 & 0.0035\ to\ 0.0037 \\ -0.2245\ to - 0.2242 & 0.9733\ to\ 0.9734 & 0.04185\ to\ 0.4184 \\ 0.0005\ to\ 0.0087 & -0.0411\ to - 0.0410 & 0.9986\ to\ 0.9989 \end{pmatrix} \tag{10}$$

This calculation is consistent with CKMfitter experimental data.

4. CALCULATIONS OF ANGLES AND SIDES OF UNITARITY TRIANGLE

The unitarity of the CKM-matrix implies various relations between its elements. In particular we have

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$

The above relation can be represented as a "unitarity" triangle in the complex $(\bar{\rho}, \bar{\eta}\ )$ plane.

To determine the length of the sides of the unitarity triangle in (Fig: 1) we use the formulae involving the wolfenstein prameters.

$$CA = \sqrt{\bar{\rho}^2 + \bar{\eta}^2} = 0.3572\ to\ 0.3586$$

$$BA = \sqrt{(1-\bar{\rho})^2 + \bar{\eta}^2} = 0.9233\ to\ 0.9293$$

We can easily calculate the angles of the unitary triangle by the following formulae.

$$\alpha \equiv arg\left(\frac{V_{td}V_{tb}^*}{V_{ub}V_{ud}^*}\right), \beta \equiv arg\left(\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \Upsilon \equiv arg\left(\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right)$$

So, when $V_{td} = 0.0005$ and $V_{ub} = 0.0035$ also $V_{tb}^* = 0.9986$ and $V_{ud}^* = 0.9741$,

$$\alpha = arg\left(\frac{0.0005 \times 0.9986}{0.0035\ \times\ 0.9741}\right) = 0.15$$

Also, when $V_{td} = 0.0087$ and $V_{ub} = 0.0037$ also $V_{tb}^* = 0.9989$ and $V_{ud}^* = 0.9744$

$$\alpha = arg\left(\frac{0.0087 \times 0.9989}{0.0035 \times\ 0.9744}\right) = 2.54$$

So, $\alpha = 0.15\ to\ 2.54$

Likewise, we get $\beta = 0.13\ to\ 1.6$ and $\Upsilon = 0.8\ to\ 1.4$

We also have,

$$sin(2\beta) = \frac{2\bar{\eta}(1-\bar{\rho})}{(1-\bar{\rho})^2 + \bar{\eta}^2}$$

Putting the values of the four parameters we get the value as

$$sin(2\beta) = 0.66\ ^{o} to\ 0.67^{o}.$$

5. RESULTS AND DISCUSSION

We can see that our calculations of the elements of the CKM matrix given in eq. (10), and the angles and the sides of the unitarity triangles are within the experimental errors. In other words, our calculated results are fairly in agreement with the experimental data. Not all the observables in flavour physics can be used as inputs to constrain the CKM matrix, due to limitations on our experimental and/or theoretical knowledge on these quantities. The list of inputs to the global fit fulfills the double requirement of a satisfying control of the attached theoretical uncertainties and a good experimental accuracy of their measurements. In addition, we only take as inputs the quantities that provide constraints on the CKM parameters $A, \lambda, \overline{\rho}$ and $\overline{\eta}$. Not all parameters are equally relevant for the global fit. In this work we see the status of the global fit of the CKM parameters within the Standard Model performed by the CKMfitter group.

Currently, all measurements of the unitary triangle are consistent with the peak lying somewhere within the red outlined region. By improving the current measurements and performing new ones, scientists will reduce the size of this allowed region, measuring the position of the vertex ever more precisely. If any experimental result is inconsistent with this vertex, scientists have evidence that the CKM picture of the weak force is incomplete. Such a discovery would overhaul our current understanding of the weak force, and provide us glimpse of new physics that may have played a role in the evolution of the early universe. In extension of this work, one can further study CP violation the kaon decay system, B-factories, and verify the results of LHCb data by minimal flavor violation theory.

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